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ON COMPARATIVE DYNAMICS

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Lately, there has been an increased interest in stability of growth paths, see, e.g., Brock and Schesinkman [3]. The problem has been stated in terms of properties of stationary paths. In order to appreciate the difficulty of the general stability problem one must realize that there are two types of "time" involved in the analysis; stability "time" and path "time." Thus, the appropriate mathematical field is that of differential equations defined on a space of functions rather than a finite dimensional space. Naturally, if one restricts one's attention to stationary paths then the usual stability analysis is appropriate. However, we would be then discussing the asymptotic behavior of the asymptotic state of the economy. This note strives to put the problem of path stability in the proper perspective by discussing the much simpler problem of comparative dynamics. Unfortunately this term has been used in the economic growth literature to discuss the basically comparative statics problem of comparing stationary growth paths. By comparative dynamics we mean the determination of the "direction" of change in the optimal path of decision variables due to a change in the exogenous variables.

The traditional method of deriving comparative statics results has been to use second order conditions for optimality. However, if one is willing to assume concavity, these results could be derived in a more direct way by utilizing the fact that a differentiable concave function lies below its tangent plane. We shall use this concept in deriving the main inequalities of this note. By way of motivation, we first derive two inequalities of comparative statics. Then we derive the comparative dynamics results and finally we discuss some economic theoretical examples.

1. <u>Comparative Statics</u>:

Consider the problem of maximizing $f(z;b)$ subject to $g(z;b) \geqq 0$ where $x \in E^n$ is the decision vector, $b \in E^\ell$ is a vector of parameters, f is real valued and where g is m-valued. Let $\hat{x}^1$ and $\hat{x}^2$ denote solutions to the problem corresponding to the values b^1 and b^2 of the parameters respectively. Let $f^{11} = f(\hat{x}^1, b^1)$, $f^{12} = f(\hat{x}^1, b^2)$, $f^{21} = f(\hat{x}^2, b^1)$ and $f^{22} = f(\hat{x}^2, b^2)$. Define the Lagrangian $L = f + \mu g$. Define g^{11}, g^{12}, g^{21}, g^{22} as we defined f^{ij} $i = 1, 2$. Finally define $L^{11} = f^{11} + \mu^1 g^{11}$ and $L^{22} = f^{22} + \mu^2 g^{22}$. Assuming some form of the constraint qualification and that f and g are differentiable we have the first order necessary conditions:

1.a) L^{11} and L^{22} are well defined, μ^1 and μ^2 are non-negative vectors and $\mu^1 g^{11} = 0$, $\mu^2 g^{22} = 0$.

1.b) $L_x^{11} = 0$ and $L_x^{22} = 0$, where $L_x^{jj} = (L_{x_1}^{jj}, \ldots, L_{x_n}^{jj})$, $j = 1, 2$ and where $L_{x_i}^{jj} = \left.\frac{\partial L}{x_i}\right|_{\substack{x = \hat{x}^j \\ b = b^j \\ \mu = \mu^j}}$

Now suppose f and g are concave functions in x for any fixed b. Then, by concavity and differentiability:

2.i) $f^{22} - f^{11} = f^{22} - f^{21} + f^{21} - f^{11} \le f^{22} - f^{21} + f_x^{11} \Delta x$, where $\Delta x = \hat{x}^2 - \hat{x}^1$.

Further, by concavity and differentiability of g and by (1.a) we have

2.ii) $0 \le \mu^1 g^{22} - \mu^1 g^{11}$

$$= \mu^1 g^{22} - \mu^1 g^{21} + \mu^1 g^{21} - \mu^1 g^{11} \le \mu^1 g^{22} - \mu^1 g^{21} + \mu^1 g_x^{11} \Delta x$$

Adding 2.i and 2.ii and using 1.b we have:

3.i) $f^{22} - f^{11} \le f^{22} - f^{21} + \mu^1 g^{22} - \mu^1 g^{21}$

Similarily we get

3.ii) $f^{11} - f^{22} \le f^{11} - f^{12} + \mu^2 g^{11} - \mu^2 g^{12}$

Adding the last two inequalities we have:

4) $(f^{22} - f^{21}) + (f^{11} - f^{12}) + (\mu^1 g^{22} - \mu^1 g^{21}) + (\mu^2 g^{11} - \mu^2 g^{12}) \geq 0.$

Now suppose <u>f</u> and <u>g</u> <u>are concave in b for any fixed x,</u>* then by (3.i) and (3.ii) we have, with $\Delta b = b^2 - b^1$,

5.i) $f^{22} - f^{11} \le f_b^{21} \Delta b + \mu^1 g_b^{21} \Delta b$

5.ii) $f^{11} - f^{22} \le - f_b^{12} \Delta b - \mu^2 g_b^{12} \Delta b.$

Adding the two last inequalities we have:

6) $(f_b^{21} - f_b^{12}) \Delta b + (\mu^1 g_b^{12}) \Delta b \geq 0$

Depending on the type of problem at hand, one may choose to apply formula 3, 4, 5 or 6. For instance, if the maximand is independent of b, 3 reduces to

3'.i) $f^{22} - f^{11} \le \mu^1 g^{22} - \mu^1 g^{21}$

2'.ii) $f^{11} - f^{22} \le \mu^2 g^{11} - \mu^2 g^{21}$

These formulae may be used to derive Hicks' generalized law of

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* In that case we say that f and g are biconcave in x and b in the sense of being concave in x for fixed b and in b for fixed x.

demand in consumer theory. On the other hand, if the constraints are independent of the parameters, relation 6 becomes:

6') $(f_b^{21} - f_b^{12})\,\Delta b \geq 0.$

This relation would determine for this case, the effect of a change in the parameter b if the objective function is biconcave.

There are many applications of these relations to economic theory and to various areas of operations research. We shall not expound on these applications here.

2. Comparative Dynamics, Discrete Time:

A direct application of the results of the previous section is possible in the case of problems of mathematical programming over discrete time with a finite horizon. We shall limit our attention to so called optimal control problems of a discrete time variety. We have two objectives in mind: 1) to anticipate results for continuous time and 2) to generalize some optimal growth results to the case of general optimality criteria, i.e., to the case where the criterion function is not the sum of "instantaneous" criterion functions.

Let Z_t denote state variables, where Z_t is an n-vector and where $t = 0, 1, \ldots, T$ and T is finite. Let U_t be the control, m-vector. Suppose the "system" is given by

7) $Z_t - Z_{t-1} \leq f^t\,(Z_{t-1}, U_t; b_t, B) \qquad t = 1, \ldots, T,$

where b_t is an ℓ-vector of "current" parameters and where B is an s-vector of "common" parameters. Suppose there are the following constraints:

8.i) $h^0(Z_0, b_0, B) \geq 0$

8.ii) $h^t(Z_{t-1}, U_t; b_t, B) \geq 0 \qquad t = 1, \ldots, T$

8.iii) $h^{T+1}(Z_T, b_{T+1}, B) \geq 0$

where h^k has r_k components,* $k = 0, 1, \ldots, T$. The objective is to maximize the function $\emptyset(Z_0, Z_1, \ldots, Z_T, U_1, \ldots, U_T; b_0, b_1, \ldots, b_T, b_{T+1}, B)$. We introduce the notation: $\zeta = (Z_0, Z_1, \ldots, Z_T)$, $\eta = (U_1, \ldots, U_T)$, $\beta = (b_0, b_1, \ldots, b_{T+1}, B)$.

The first order necessary conditions, assuming a constraint qualification, are the existence of non-negative vectors $\lambda^t, \mu^0, \mu^t, \mu^{T+1}$, $t = 1, \ldots, T$ such that:

9.i) $\lambda^t(f^t - (Z_t - Z_{t-1})) = 0,\ \mu^0 h^0 = 0,\ \mu^t h^t = 0,\ \mu^{T+1} h^{T+1} = 0,$

where the functions are evaluated at the optimum.

9.ii) $\lambda^t - \lambda^{t+1} = \emptyset_{Z_t} + \lambda^{t+1} f^{t+1}_{Z_t} + \mu^{t+1} h^{t+1}_{Z_t}, \qquad t = 1, \ldots, T-1$

9.iii) $\emptyset_{Z_0} + \mu^0 h^0_{Z_0} + \lambda^1 f^1_{Z_0} + \lambda^1 = 0$

9.iv) $\emptyset_{Z_T} + \mu^{T+1} h^{T+1}_{Z_T} - \lambda^T = 0$

9.v) $\emptyset_{U_t} + \lambda^t f_{U_t} + \mu^t h_{U_t} = 0.$

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* Thus we don't necessarily have the same number of constraints at each t.

In the particular case where

$$10)\qquad \emptyset = \sum_{t=1}^{T} V^t(U_t, Z_{t-1}, b_t, B) + \omega^0(Z_0, b_0, B) + \omega^T(Z_T, b_{T+1}, B)$$

we may define the functions H^t, G^0, G^T as follows:

$$H^t = V^t + \lambda^t f^t + \mu^t h^t$$

$$G^0 = \omega^0 + \mu^0 h^0$$

$$G^T = \omega^T + \mu^{T+1} h^{T+1}$$

Conditions (9) may be written as:

$$11.\ i)\qquad \lambda^{t+1} - \lambda^t = - H^{t+1}_{Z_t}$$

$$11.\ ii)\qquad H^t_{U_t} = 0$$

$$11.\ iii)\qquad G^0_{Z_0} + H^1_{Z_0} = 0$$

$$11.\ iv)\qquad G^T_{Z_T} + H^{T+1}_{Z_T} = 0.$$

These conditions are the first order necessary conditions for the "usual" optimal control problem in discrete time.

As we did in the last section we shall consider the effect of changes in the parameters b_0, b_t, b_{T+1}, B. Rather than derive the formulae again, we could obtain them for this problem by directly applying the results of the last section. First we write down the general formulae, then we derive some special results.

What corresponds to formula (3) would be:

$$12.\ i)\qquad \emptyset^{22} - \emptyset^{11} \leq \emptyset^{22} - \emptyset^{21} + \sum_{t=1}^{T} \lambda^{t,1}(f^{t,22} - f^{t,21}) + \sum_{\tau=0}^{T+1} \mu^{\tau,1}(h^{\tau,22} - h^{\tau,21})$$

$$12.\ ii)\qquad \emptyset^{11} - \emptyset^{22} \leq \emptyset^{11} - \emptyset^{12} + \sum_{t=1}^{T} \lambda^{t,2}(f^{t,11} - f^{t,12}) + \sum_{\tau=0}^{T+1} \mu^{\tau,2}(h^{\tau,11} - h^{\tau,12}),$$

where $h^{\tau,kk'} = h(\zeta^k, \eta^k; \beta^{k'})$, $kk' = 1, 2$; $\tau = 0, 1, \ldots, T+1$ and where $f^{t,kk'}$ is defined in the same way. Note that formula (12) assumes concavity of f and h in ζ and η and follows immediately from (3). If we also assume concavity in β, for fixed ζ and η, we have

$$13.\ i)\qquad \emptyset^{22} - \emptyset^{11} \leq \emptyset^{21}_{\beta}\Delta\beta + \sum_{t=1}^{T} \lambda^{t,1} f^{t,21}_{\beta}\Delta\beta + \sum_{\tau=0}^{T+1} \mu^{\tau,1} h^{\tau,21}_{\beta}\Delta\beta.$$

$$13.\ ii)\qquad \emptyset^{11} - \emptyset^{22} \leq - \emptyset^{12}_{\beta}\Delta\beta \sum_{t=1}^{T} \lambda^{t,2} f^{t,12}_{\beta}\Delta\beta - \sum_{\tau=0}^{T+1} \mu^{\tau,2} h^{\tau,12}_{\beta}\Delta\beta.$$

$$14)\qquad (\emptyset^{21}_{\beta} - \emptyset^{12}_{\beta})\Delta\beta + \sum_{t=1}^{T} \lambda^{t,2}(f^{t,21} - f^{t,12})\Delta\beta + \sum_{\tau=0}^{T+1}(\mu^{\tau,1} h^{\tau,21}_{\beta} - \mu^{\tau,2} h^{12}_{\beta})\Delta\beta \geq 0.$$

Formulae (13) and (14) follow directly from (5) and (6) of the last section.

As our first economic example, we take the problem of a consumer who plans his consumption over a period of T intervals. Let C_t denote consumption at time t and let $C = (C_1, \ldots, C_T)$. Let P_t denote the price vector at time t, θ_t denote the interest rate at time t and let W_t be the wealth at time t. Then we have:

$$15)\qquad W_t - W_{t-1} \leq M_t + \theta_t W_{t-1} - P_t C_t,\ W_0 \leq S_0,$$

where M_t is current income and where S_0 is given. Suppose the consumers' preferences are given by the function: $\emptyset(C)$, and suppose he has a certain "objective" to be attained at the end of the plan, expressed by: $W_T - \delta_T \geq 0$. The consumers' problem may be stated as follows: maximize $(\emptyset(C))$ subject to (15), $C_t \geq 0$ $W_t \geq 0$ and $W_T - \delta_T \geq 0$. Our non-negativity constraint on wealth rules out being in debt at any time.

The first order necessary conditions may be derived by applying (9) above. They are:

16.i) $\lambda^t(M_t + \theta_t W_{t-1} - P_t C_t - (W_t - W_{t-1})) = 0, \quad \mu_1^0(S_0 - W_0) = 0,$

$$\mu_1^t W_t = 0, \quad \mu_1^{T+1}(W_T - \delta_T) = 0, \quad \mu_2^t C_t = 0.$$

16.ii) $\lambda^t - \lambda^{t+1} = \theta_t \lambda^{t+1} + \mu_1^{t+1} \qquad t = 1, \ldots, T-1$

16.iii) $\lambda^1(1 + \theta_1) - \mu_1^0 = 0$

16.iv) $\mu_1^{T+1} - \lambda^T = 0$

16.v) $\emptyset_{C_t} - \lambda^t P_t + \mu_2^t = 0$

Assuming that $W_t > 0$ and that $C_t > 0$ ($\mu_1^t = 0$, $\mu_2^t = 0$, $t = 1, \ldots, T$) we have:

$$\emptyset_{C_1} = \frac{\mu_1^0}{(1 + \theta_1)} P_1$$

$$\emptyset_{C_t} = \frac{\mu_1^0}{(1 + \theta_1)} \prod_{\tau=1}^{t-1} (1 + \theta_\tau)^{-1} P_t \qquad t = 2, \ldots, T.$$

In particular we have the following expression of the "intertemporal marginal rate of substitution";

17) $$\frac{\emptyset_{C_{t,j}}}{\emptyset_{C_{t+1,j}}} = \frac{P_{tj}(1 + \theta_t)}{P_{t+1,j}},$$

where $C_{t,j}$ is the j^{th} component of C_t.

This last equation is well known but it is typically derived only for utility functions in the form $\emptyset(C) = \sum_t V^t(C_t)$. To illustrate the application of our formulae, we shall further specialize the form of $\emptyset$ to be:

18) $$\emptyset(C_t) = \sum_{t=1}^{T} V(C_t)(1 + \rho)^{-t} \; *$$

Then, (17) becomes

$$\frac{V_{C_{t,j}}}{V_{C_{t+1,j}}} = \frac{P_{tj}}{P_{t+1,j}} \left(\frac{1 + \theta_t}{1 + \rho}\right).$$

Hence with constant prices over time, consumption is higher in the future if the personal rate of time discount exceeds the market rate of interest, and conversely.

We consider the effect of a "compensated" change in θ, assuming a rate of interest that is constant over time and assuming that compensation means returning the consumer to the previous utility level. Utilizing (13) we have

19.i) $$0 \leq \sum_{t=1}^{T} \lambda^{t,1} W_{t-1}^2 \Delta\theta$$

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* Note that in this formulation of $\emptyset$, concavity of $\emptyset$ is equivalent to concavity (or quasi-concavity) of V.

19. ii) $$0 \leq -\sum_{t=1}^{T} \lambda^{t,2} W^{1}_{t-1} \Delta\theta$$

Now $$\lambda^{t,1} = \frac{\mu^{0,1}}{(1+\theta^{1})} \prod_{\tau=1}^{t-1} (1+\theta^{1})^{-\tau} = \frac{\mu^{0,1}}{(1+\theta^{1})} (1+\theta^{1})^{1-t} = \mu^{1}_{0}(1+\theta^{1})^{-t}$$

$t = 1, \ldots, T.$

and $$\lambda^{t,2} = \mu^{0,2}(1+\theta^{2})^{-t} \qquad t = 1, \ldots, T.$$

If

20. i) $$\mu^{0,1} > 0, \ \mu^{0,2} > 0$$

so that $W_0 = S_0$, then

20. ii) $$\sum_{t=1}^{T} (1+\theta^{1})^{-t} W^{2}_{t} \Delta\theta \geq 0$$

$$\sum_{t=1}^{T} (1+\theta^{2})^{-t} W^{1}_{t} \Delta\theta \geq 0 \ .$$

Adding the inequalities of (20) we have:

$$\Delta\theta \sum_{T=1}^{T} (1+\theta^{1})^{-t} \left[W^{2}_{t} - \left[\frac{(1+\theta^{1})}{(1+\theta^{2})} \right]^{t} W^{1}_{t} \right] \geq 0$$

In the time t, if $\Delta\theta$ is very small, this reduces to:

$$d\theta \sum_{t} (1+\theta^{1})^{-t} dW_{t} \geq 0 \ .$$

Thus the present value of the changes in the optimal wealth changes in the same direction as the rate of interest. Now let us examine the effect of a compensated change in the price of a given good, j, at a given time, t_0.

Again utilizing (13) we have:

$$0 \leq +\lambda^{t_0,1} C^{2}_{t_{0,j}} \Delta P_{t_{0,j}}$$

$$0 \leq -\lambda^{t_0,2} C^{1}_{t_{0,j}} \Delta P_{t_{0,j}}$$

Assuming $\lambda^{t_0,1}$ and $\lambda^{t_0,2}$ not to be zeroes, (the budget constraints are satisfied as equalities), we have $\Delta C_{t_{0,j}} \Delta P_{t_{0,j}} \leq 0$, since the multipliers are postives. Thus we have the familiar result about the compensated effect of own price change. Correspondence of results may be obtained about the effect of changing the income stream.

We now move to the problem of optimal economic growth. We shall illustrate the applications of our formulae by using the neoclasical model of optimal growth and specializing that to the case of discrete finite time. Let x_t denote the stock of goods at time t and let C_t be the consumption at time t. The growth of the economy may be described by:

21) $$C_t + x_t - x_{t-1} \leq f^{t}(x_{t-1}, b_t), \ C_t \geq 0, \ x_t \geq 0, \quad t = 1, \ldots, T.$$

We assume that consumption results in utility $\phi(C)$, $C = (C_1, \ldots, C_T)$, and that the initial situation in the economy is given by $h^0(x_0, b_0) \geq 0$ and that the goal set of the economy is given by $h^{T+1}(x_T, b_{T+1}) \geq 0$. The optimality problem may be stated as: max $\phi(C)$ subject to (21) and to:

22) $$h^0(x_0, b_0) \geq 0 \ , \qquad h^{T+1}(x_T, b_{T+1}) \geq 0.$$

The first order necessary conditions for this problem are:

23. i) $\lambda^t(f^t - C_t - (x_t - x_{t-1})) = 0, \quad \mu^0 h^0 = 0, \quad \mu^{T+1} h^{T+1} = 0,$

$$\mu_1^t C_t = 0, \quad \mu_2^t x_t = 0$$

23. ii) $\lambda^t - \lambda^{t+1} = \lambda^{t+1} f_{x_t}^{t+1} + \mu_2^t \qquad t = 1, \ldots, T-1$

23. iii) $\lambda^1 + \lambda^1 f_{x_0}^1 + \mu^0 h_{x_0}^0 = 0$

23. iv) $\mu^{T+1} h_{x_T}^{T+1} - \lambda^T = 0$

23. v) $\emptyset_{C_t} - \lambda^t + \mu_1^t = 0$

These conditions have the usual myopic interpretation when $\emptyset$ is expressible in the form $\emptyset = \sum_j V^t(C_t)$. We wish to study the effect of a change in the terminal goal set. Assume the end point condition $h^{T+1} \geq 0$ is given by the inequality: $X_T \geq b_{T+1}$ and suppose the initial condition is given by: $X_0 \leq b_0$. Suppose there is a change in b_{T+1}; then the effect may be discussed by utilizing relations (13) and (23). We have:

24. i) $\emptyset^{22} - \emptyset^{11} \leq -\mu^{T+1,1} \Delta b_{T+1}$

24. ii) $\emptyset^{11} - \emptyset^{22} \leq \mu^{T+1,2} \Delta b_{T+1}$.

We note the interesting implication of (24. i) that a rise in the target stocks will result in a lower utility level at the new equilibrium. This is, of course, due to the fact that the model assigns no utility to stocks after the plan is over. By the first order conditions, $\mu^{T+1} = \lambda^T$ and thus:

25. i) $\mu^{0,1} \prod_{\tau=1}^{T-1} (1 + f_x^{\tau,11})^{-1} \Delta b_{T+1} \geq \emptyset^{22} - \emptyset^{11}$

25. ii) $-\mu^{0,2} \prod_{\tau=1}^{T-1} (1 + f_x^{\tau,22})^{-1} \Delta b_{T+1} \geq \emptyset^{11} - \emptyset^{22}$

Suppose now that the change in b_{T+1} is "compensated" so that $\emptyset^{22} = \emptyset^{11}$. Then, since $\mu^{0,1} \geq 0$ and $\mu^{0,2} \geq$ we have:

26) $$\Delta b_{T+1} \left[\prod_{\tau=1}^{T-1} (1 + f_x^{\tau,11})^{-1} - \prod_{\tau=1}^{T-1} (1 + f_x^{\tau,22})^{-1} \right] \geq 0 \ .$$

3. Comparative Dynamics, Continuous Time:

In this section we use a technique of proof that was used by Magasarian [2] to derive some inequalities concerning the effect of a change in the exogenous variables on the solutions of an optimal control problem. The solution of our control problem depends on the vaues of some exogenous variables, i. e., variables that are not determined by the solution of the problem. Once these variables are given, the problem may be solved in the usual ways. Rather than solve for the optimal controls and state variables in terms of the exogenous variables, it might be of interest to obtain pairwise comparisons of optimal decisions under pairs of "values" of the exogenous variables.

Let a system be given by:

27) $\dot{x}_t = f(t, x_t, u_t, \beta_t, B)$,

here $x \in E^{n_1}$, $u \in E^{n_2}$, $\beta \in E^{n_3}$, $B \in E^{n_4}$, and where β_t and B are not choice variables. Suppose the control constraints are given by:

28) $\quad h^{\alpha}(t, x_t, u_t, \beta_t, B) \geq 0, \qquad \alpha = 1, \ldots, \overline{\alpha}$.

Let the end point conditions be given by:

29) $\quad g^{\delta}(x_{t_0}, x_{t_1}) \geq 0, \qquad \delta = 1, \ldots, \overline{\delta}$.

The objective of the system is to maximize:

$$w(I, x_{t_0}, x_{t_1}, B),$$

where

30) $\quad I = \int_{t_0}^{t_1} \phi(t, x_t, u_t, \beta_t, B)$.

Assume the functions and admissible controls satisfy all of the regularity conditions that assure that the first order necessary condition theorems apply, e.g., those given in Hestenes [1]. Let (β_t^1, B^1) and (β_t^2, B^2) be two values of the exogenous variables. Then we would have two, not necessarily distinct, solutions $z^1 = \left(x_t^1, u_t^1, x_{t_0}^1, x_{t_1}^1\right)$ and $z^2 = \left(x_t^2, u_t^2, x_{t_0}^2, x_{t_1}^2\right)$ of the control problem. We use the superscript ij on a function to indicate that it is evaluated at the value i of the variables z and the value j for the variables (β_t, B), $i, j = 1, 2$. We use the subscripts z, β_t, B to indicate differentiation with respect to these variables. The first order conditions for optimal solutions are the existence of time functions $\lambda^i(t)$ and $\mu^i(t)$ and of constants λ_0^i, γ^i, where $\lambda^i(t)$ has values in E^n, $\mu^i(t)$ has values in E^{α}, $\mu_0^i \geq 0$ is a scalar

and $\gamma^i \in E^{\overline{\delta}}$, such that, if $H = \lambda_0 w + \lambda f + \mu h$

31) $\quad \dot{\lambda}^i = - H_x^{ii}, \quad i = 1, 2, \quad$ that is, $\dot{\lambda}^i = - \left[\lambda_0^i w_I^{ii} \phi_x^{ii} + \lambda^i f_x^{ii} + \mu^i h_x^{ii}\right]$

32) $\quad \mu^i \geqq 0, \quad \mu^i h^{ii} = 0, \quad i = 1, 2$.

33) $\quad \gamma^i \geqq 0, \quad \gamma^i g^i = 0, \quad i = 1, 2$.

34) $\quad H_u^{ii} = 0, \quad i = 1, 2, \quad$ that is, $\lambda_0 w_I^{ii} \phi_u^{ii} + \lambda^i f_u^{ii} + \mu^i h_u^{ii} = 0$.

35) $\quad \lambda^i(t_0) + G_{x_{t_0}}^{ii} = 0, \quad i = 1, 2, \quad$ where $G = \lambda_0 w + \gamma g$.

36) $\quad \lambda^i(t_1) - G_{x_{t_1}}^{ii} = 0$.

Assume that w, f, h and g are concave in z for fixed (β_t, B), are differentiable and that either $\lambda^i \geqq 0$, $i = 1, 2$, or f is linear. Then the above conditions are also sufficient for a maximum if $\lambda_0 > 0$, as could be easily seen by using Mangasarian's methods [2]. From the concavity of w it follows that:

37) $$w^{22} - w^{11} = w^{22} - w^{21} + w^{21} - w^{11} \leq w^{22} - w^{21} + w_I^{11} \Delta I + w_{x_{t_0}} \Delta x_{t_0} + w_{x_{t_1}} \Delta x_{t_1},$$

where $\Delta I = \int_{t_0}^{t_1} (\phi^{21} - \phi^{11}) dt$ and $\Delta z = z^2 - z^1$. Relation (37), assuming $w_I \geq 0$, may be written as:

37.i) $$w^{22} - w^{11} \leq w^{22} - w^{21} + w_I^{11}\left(\int_{t_0}^{t_1} \left[\phi_x^{11} \Delta x + \phi_u^{11} \Delta u\right] dt\right) + w_{x_{t_0}} \Delta x_{t_0} + w_{x_{t_1}} \Delta x_{t_1} .$$

With $\lambda^1 \geqq 0$, f concave in z (or f linear in z), we have

38) $$\lambda^1 \Delta \dot{x} = \lambda^1 (f^{22} - f^{11}) \leqq \lambda^1 (f^{22} - f^{21}) + \lambda^1 f_x^{11} \Delta x + \lambda^1 f_u^{11} \Delta u .$$

Since h and g are concave in z, we have, using (32) and (33):

39) $$0 \leq \mu^1 (h^{22} - h^{11}) \leq \mu^1 (h^{22} - h^{21}) + \mu^1 h_x^{11} \Delta x + \mu^1 h_u^{11} \Delta u$$

40) $$0 \leq \gamma^1 (g^{22} - g^{11}) \leq \gamma^1 (g^{22} - g^{21}) + \gamma^1 g_{x_{t_0}}^{11} \Delta x + \gamma^1 g_{x_{t_1}}^{11} \Delta x_{t_1} .$$

Multiplying both sides of (37. i) by λ_0, integrating both sides of (38) and (39), and adding the results to (40) we get, in view of (31), (34), (35) and (36):

41) $$\lambda_0^1 (w^{22} - w^{11}) + \int_{t_0}^{t_1} \lambda^1 \Delta \dot{x}\, dt \leq \lambda_0^1 (w^{22} - w^{21}) + \gamma^1 (g^{22} - g^{21})$$

$$+ \lambda^1 (t_1) \Delta_{t_1} - \lambda^1 (t_0) \Delta x_{t_0} - \int_{t_0}^{t_1} \dot{\lambda}^1 \Delta x\, dt$$

$$+ \int_{t_0}^{t_1} \left[\lambda^1 (f^{22} - f^{21}) + \mu^1 (h^{22} - h^{21}) \right] dt.$$

Thus, since $\int_{t_0}^{t_1} (\lambda^1 \Delta \dot{x} + \dot{\lambda}^1 \Delta x) dt = \lambda^1 (t_1) \Delta x_{t_1} - \lambda^1 (t_0) \Delta x_{t_0}$, we have

42) $$\lambda_0^1 (w^{22} - w^{11}) \leq \lambda_0^1 (w^{22} - w^{21}) + \gamma^1 (g^{22} - g^{21})$$

$$+ \int_{t_0}^{t_1} \left[\lambda^1 (f^{22} - f^{21}) + \mu^1 (h^{22} - h^{21}) \right] dt .$$

Assuming concavity in (β_t, B) for any fixed z and denoting $\beta_t^2 - \beta_t^1$ by $\Delta\beta$ and $B^2 - B^1$ by ΔB we have:

43) $$\lambda_0^1 (w^{22} - w^{11})$$

$$\leq \int_{t_0}^{t_1} \left\{ \left[\lambda_0^1 w_I^{21} \phi_\beta^{21} + \lambda^1 f_\beta^{21} + \mu^1 h_\beta^{21} \right] \Delta\beta + \left[\lambda_0^1 w_I^{21} \phi_B^{21} + \lambda^1 f_B^{21} + \mu^1 h_B^{21} \right] \Delta B \right\} dt$$

$$+ (\lambda_0^1 w_B^{21} + \gamma^1 g_B^{21}) \Delta B$$

Repeating the process leading to (42) we have:

44) $$\lambda_0^2 (w^{11} - w^{22}) \leq \lambda_0^2 (w^{11} - w^{12}) + \lambda^2 (g^{11} - g^{12})$$

$$+ \int_{t_0}^{t_1} \left[\lambda^2 (f^{11} - f^{12}) + \mu^2 (h^{11} - h^{12}) \right] dt.$$

Assuming concavity in β_t and B for any fixed z we have:

45) $$\lambda_0^2 (w^{11} - w^{22})$$

$$\leq \int_{t_0}^{t_1} \left\{ \left[\lambda_0^2 w_I^{12} \phi_\beta^{12} + \lambda^2 f_\beta^{12} + \mu^2 h_\beta^{12} \right] (-\Delta\beta) + \left[\lambda_0^2 w_I^{12} \phi^{12} + \lambda^2 f_B^{12} + \mu^2 h_B^{12} \right] (-\Delta B) \right\} dt$$

$$+ (\lambda_0^2 w^{12} + \gamma^2 g^{12})(-\Delta B) .$$

The inequalities (42) - (45) were derived without assuming λ_0^i to be non-zero. <u>If we assume that $\lambda_0^i > 0$</u> then λ_0^i may be set equal to unity and we have some further results. Adding (42) and (44) we have:

46) $$(w^{22} - w^{21}) + (w^{11} - w^{12}) + \gamma^1 (g^{22} - g^{21}) + \gamma^2 (g^{11} - g^{12})$$

$$+ \int_{t_0}^{t_1} \left[\lambda^1 (f^{22} - f^{21}) + \lambda^2 (f^{11} - f^{12}) + \mu^1 (h^{22} - h^{21}) + \mu^2 (h^{11} - h^{12}) \right] dt \geq 0 .$$

We also get from (43) and (45):

$$47)\quad \int_{t_0}^{t_1}\Big\{\Big[(w_I^{21}\phi_\beta^{21} - w_I^{12}\phi_\beta^{12}) + (\lambda^1 f_\beta^{21} - \lambda^2 f_\beta^{12}) + (\lambda^1 h_\beta^{21} - \lambda^2 h_\beta^{12})\Big]\Delta\beta$$

$$+ \Big[(w_I^{21}\phi_B^{21} - w_2^{1}\phi_B^{12}) + (\lambda^1 f_B^{21} - \lambda^2 g_B^{12}) + (\lambda^1 h_B^{21} - \lambda^2 g_B^{12})\Big]\Delta B\Big\}dt$$

$$+ \Big[(w_B^{21} - w_B^{12}) + (\gamma^1 g_B^{21} - \gamma^2 g_B^{12})\Big]\Delta B \geq 0 \ .$$

4. Some Economic Examples

First we discuss the problem of a consumer maximizing a time linear utility function:

Let C(t) be the consumption at time t and let the instantaneous utility function be given by $V(C_t)$ and assume the utility function is given by: $\int_{t_0}^{T} V(C_t)e^{-\rho t}dt$. Let W_t be the wealth of the consumer, let θ be the interest rate and let P_t be the price vector. Let W_0 be given and let W_T be restricted to $W_T \geq B$ where $B > 0$. The problem is to maximize $\int_{t_0}^{T} V(C_t)e^{-\rho t}dt$ subject to

$$48.\text{i})\quad \dot{W}_t = \theta W_t + y_t - P_t C_t \ ,$$

W_0 given and y_t is a given income flow.

$$48.\text{ii})\quad W_T \geq B \ .$$

The necessary conditions are:

$$49.\text{i})\quad \dot{\lambda}_t = -\theta\lambda_t,\ \ V_{C_t}e^{-\rho t} = \lambda_t P_t,\ \ \lambda_T = \gamma^T \qquad \gamma^T(W_T - B) = 0 \ .$$

Thus, solving for λ, we have:

$$49.\text{ii})\quad V_{C_t} = P_t \gamma^T e^{\theta T + (\rho - \theta)t} \ .$$

Thus the time path of consumption is directly related to the difference between the personal rate of time preference ρ and the market rate of interest θ. If we consider two goods I and j at the same time t we have:

$$49.\text{iii})\quad V_{C_{t,i}}/V_{C_{t,j}} = P_{t,i}/P_{t,j}$$

which is the usual myopic expected result. Now suppose there is a compensated change in (P, θ) where compensation means that the consumer stays at the same utility level. By (43) we have:

$$0 \leq \int_{t_0}^{T}\Big[\lambda^1(W^2\Delta\theta - C^2\Delta P_t)\Big]dt \ .$$

Thus, since $\lambda^1 \geq 0$,

$$\int_{t_0}^{T}\Big[W^2\Delta\theta - C^2\Delta P_t\Big]dt \geq 0 \ .$$

Similarly, using (45)

$$\int_{t_0}^{T}\Big[-W^1\Delta\theta + C^1\Delta P_t\Big]dt \geq 0 \ ,$$

since $\lambda^2 \geq 0$. Thus,

$$50)\quad \int_{t_0}^{T}\Big[\Delta W\Delta\theta - \Delta C\Delta P_t\Big]dt \geq 0 \ .$$

Relation (50) is an integrated form of the generalized law of demand and which may be used to derive the appropriate "Slutsky" conditions. We have (50) in the cumulative form because of the form of the total utility function and because of our method of compensation.

Next, we discuss a simple one sector model of optimal growth. Let the accumulation equation be given by

51) $\dot{K}_t = I_t - \alpha K_t$,

whose K_t is capital stock and where I_t is gross investment and α is the rate of depreciation. Let C_t be consumption and let production be given by $f(K_t, L_t; b_t)$ where L_t is labor fully employed and where b_t is a shift parameter. The economy aspires to a goal given by $f(K_T, L_T, b_T) \geq B$. The goal expresses a certain potential for the economy. The objective is to maximize a function $\int_{t_0}^{T} V(C_t)e^{-\rho t}$ subject to feasibility, i.e., subject to (51) and

52) $f - C_t - I_t \geq 0$. $\quad C_t \geq 0, I_t \geq 0$.

Let $H = Ve^{-\rho t} + \lambda(I - \alpha K) + \mu(f - C - I)$. The first order conditions for an interior solution are given by:

53.i) $\dot{\lambda} = \alpha\lambda - \mu f_K$

53.ii) $V_C e^{-\rho t} = \mu$

53.iii) $\lambda = \mu \qquad \lambda(T) = \gamma^T f_{K_T}$.

We discuss the effect of a shift in L (which may be taken as a shift in the population pattern) a shift in b (which could represent technological change), a change in B which represents a change in the goal set for the economy and finally a change in ρ which is a change in the time preference of the society.

First we consider the result of a shift in L. Using formula (47) and assuming f is concave, we have

54) $\int_{t_0}^{T} (\mu^1 f_L^{21} - \mu^2 f_L^{12}) \Delta L \, dt \geq 0$.

Since μ^1 and μ^2 express shadow prices of outputs, relation (54) expresses the "total" effect of a shift in L in terms of value of the marginal product of labor.

If b changes then the effect is given, using (47), by:

55) $\int_{t_0}^{T} (\mu^1 f_b^{21} - \mu^2 f_b^{12}) \Delta b \, dt \geq 0$.

If ρ changes then the effect is given by:

56) $\int_{t_0}^{T} \left[V_C^1 \rho^2 e^{-\rho^2 t} - V_C^2 \rho^1 e^{-\rho_1 t} \right] \Delta\rho \, dt \geq 0$.

Finally, if B changes then the effect is given by:

$(\gamma^{T,2} - \gamma^{T,1}) \Delta B \geq 0$.

But, by the first order conditions this means:

$$\left(\frac{\lambda^2(T)}{f_{K_T}^{22}} - \frac{\lambda^1(T)}{f_{K_T}^{11}} \right) \Delta B \geq 0$$

substituting for λ from the first order conditions we have:

57) $$\left(\frac{V_C(C_T^2) e^{-\rho^2 T}}{f_{K_T}^{22}} - \frac{V_C(C_T^1) e^{-\rho^1 T}}{f_{K_T}^{11}} \right) \Delta B \geq 0 .$$

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